Continuity Exercises

1. Analyze whether the following function is continuous on $\mathbb{R}$. If the function is discontinuous at any point, indicate the type of discontinuity. In the case of removable discontinuity, extend the function to make it continuous at that point.

$$f(x) = \frac{x+3}{3x^2+x^3} + 1$$

2. Given $f(x) = \begin{cases} 2x^2+4x & \text{if } x \geq 1 \\ a-x & \text{if } x < 1 \end{cases}$

   (a) Indicate the value of $a \in \mathbb{R}$ for which the function is continuous on $\mathbb{R}$.

   (b) Calculate $\lim_{x\to+\infty} \dfrac{f(x)}{x^3+1}$

Solutions

1. The function has two discontinuities at $x = 0$ and $x = -3$. The domain includes all real numbers except $x = 0$ and $x = -3$. If we directly compute the limit as x approaches -3, the result is an indeterminate form of type $0/0$. By factoring:

$$\lim_{x \to -3} \frac{x+3}{3x^2 + x^3} + 1 = \lim_{x \to -3} \frac{x+3}{x^2(3+x)} + 1 = \lim_{x \to -3} \frac{1}{x^2} + 1 = \frac{1}{9} + 1 = \frac{10}{9}$$

On the other hand, for the case of x approaching 0:

$$\lim_{x \to 0} \frac{x+3}{3x^2 + x^3} + 1 = \lim_{x \to 0} \frac{x+3}{x^2(3+x)} + 1 = \lim_{x \to 0} \frac{1}{x^2} + 1 = \infty$$

We extend the function to make it continuous at $x = -3$:

$$f(x) = \begin{cases} \dfrac{x+3}{3x^2 + x^3} + 1 & \text{if } x \neq -3 \\ \dfrac{10}{9} & \text{if } x = -3 \end{cases}$$

In the case of $x = 0$, the discontinuity is essential.

2. (a) For the function to be continuous, $\lim_{x \to 1} f(x) = f(1)$. Since at all other points the function is continuous.

$$\lim_{x \to 1^+} 2x^2 + 4x = 6$$

$$\lim_{x \to 1^-} a - x = a - 1$$

$$f(1) = 2 * (1)^2 + 4 * (1) = 6$$

Therefore, $a - 1 = 6$, $a = 7$

$$f(x) = \begin{cases} 2x^2 + 4x & \text{if } x \geq 1 \\ 7 - x & \text{if } x < 1 \end{cases}$$

(b)

$$\lim_{x \to +\infty} \frac{f(x)}{x^3 + 1} = \lim_{x \to +\infty} \frac{2x^2 + 4x}{x^3 + 1} = \lim_{x \to +\infty} \frac{x^3(2/x + 4/x^2)}{x^3(1 + 1/x^3)} = \lim_{x \to +\infty} \frac{(2/x + 4/x^2)}{(1 + 1/x^3)} = 0$$